A Brief Review of Probability

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Outline

Expectation, Variance, and Densities

Important Distributions
- Discrete Distributions
- Continuous Distributions

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Expectation

The expected value of a random variable X is simply the weighted average of all possible values of X.

Discrete Case:

$$E(X) = \sum_i x_i P(X = x_i)$$

where $P(X = x)$ is the probability mass function (PMF).

Continuous Case:

$$E(X) = \int_{-\infty}^{\infty} x p(x) dx$$

where $p(x)$ is the probability density function (PDF).

Expectation of a Function of a Random Variable

Suppose we want to find $E[g(X)]$, where $g(X)$ is any function of X. We can simply weight the values of $g(x)$ by the PDF or PMF of X:

$$E[g(X)] = \sum_i g(x_i) P(X = x_i)$$

for discrete random variables and

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) p(x) dx$$

for continuous random variables.

This is sometimes known as the *Law of the Unconscious Statistician* (LOTUS).

Variance

The formula for the variance of a random variable is

$$\mathrm{Var}(X) = E[(X - E(X))^2]$$

We can find the variance using LOTUS, or we can simplify the formula first.

$$\begin{aligned}
\mathrm{Var}(X) &= E[(X - E(X))^2] \\
&= E[X^2 - 2XE(X) + (E(X))^2] \\
&= E(X^2) - 2E(X)E[E(X)] + E([E(X)]^2) \\
&= E(X^2) - 2[E(X)]^2 + [E(X)]^2 \\
&= \mathbf{E(X^2) - [E(X)]^2}
\end{aligned}$$

We can then find the first part with LOTUS.

Marginal, Conditional, and Joint Densities

$$
\begin{aligned}
p(x) &= \int p(x, y) dy \\
p(x, y) &= \int p(x, y, z) dz \\
p(x|y) &= \frac{p(x, y)}{p(y)} \\
p(x|y, z) &= \frac{p(x, y, z)}{p(y, z)} \\
p(x, y) &= p(x|y)p(y) \\
&= p(y|x)p(x) \\
p(x, y, z) &= p(x|y, z)p(y|z)p(z)
\end{aligned}
$$

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The Bernoulli Distribution

$Y \sim \text{Bernoulli}(\pi)$

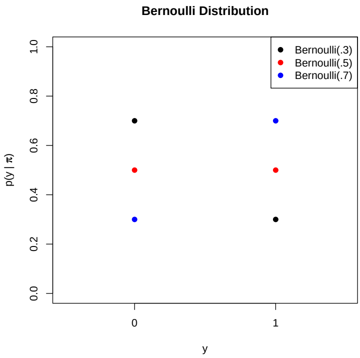
$y = 0, 1$

probability of success: $\pi \in [0, 1]$

$p(y|\pi) = \pi^y (1 - \pi)^{(1-y)}$

$E(Y) = \pi$

$\text{Var}(Y) = \pi(1 - \pi)$



The Binomial Distribution

$Y \sim \text{Binomial}(n, \pi)$

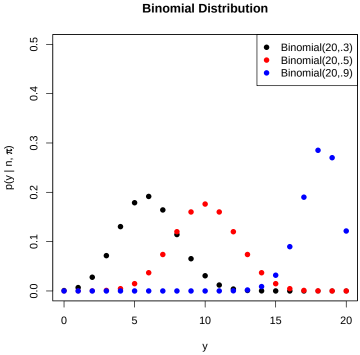
$y = 0, 1, \ldots, n$

number of trials: $n \in \{1, 2, \ldots\}$
probability of success: $\pi \in [0, 1]$

$p(y|\pi) = \binom{n}{y} \pi^y (1 - \pi)^{(n-y)}$

$E(Y) = n\pi$

$\text{Var}(Y) = n\pi(1 - \pi)$



The Multinomial Distribution

$Y \sim \text{Multinomial}(n, \pi_1, \dots, \pi_k)$

$y_j = 0, 1, \dots, n; \;\; \sum_{j=1}^{k} y_j = n$

number of trials: $n \in \{1, 2, \dots\}$
probability of success for j: $\pi_j \in [0, 1]; \;\; \sum_{j=1}^{k} \pi_j = 1$

$p(\mathbf{y}|n, \boldsymbol{\pi}) = \frac{n!}{y_1! y_2! \dots y_k!} \pi_1^{y_1} \pi_2^{y_2} \dots \pi_k^{y_k}$

$E(Y_j) = n\pi_j$

$\text{Var}(Y_j) = n\pi_j(1 - \pi_j)$

$\text{Cov}(Y_i, Y_j) = -n\pi_i\pi_j$

The Poisson Distribution

$Y \sim \text{Poisson}(\lambda)$

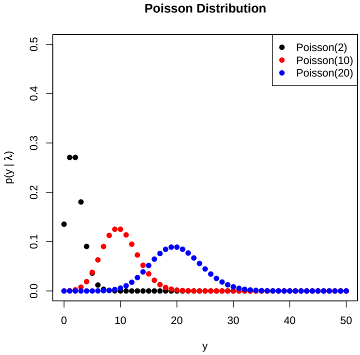
$y = 0, 1, \ldots$

expected number of occurrences: $\lambda > 0$

$p(y|\lambda) = \frac{e^{-\lambda}\lambda^{y}}{y!}$

$E(Y) = \lambda$

$\text{Var}(Y) = \lambda$



The Geometric Distribution

How many Bernoulli trials until success?

$Y \sim \text{Geometric}(\pi)$

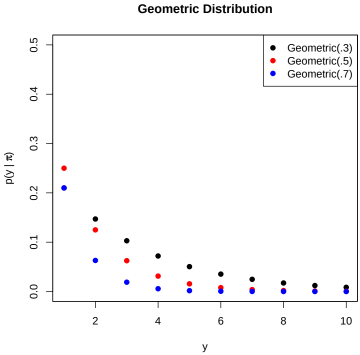
$y = 1, 2, 3, \ldots$

probability of success: $\pi \in [0, 1]$

$p(y|\pi) = (1 - \pi)^{(y-1)}\pi$

$E(Y) = \frac{1}{\pi}$

$\text{Var}(Y) = \frac{1-\pi}{\pi^2}$



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The Univariate Normal Distribution

$Y \sim \text{Normal}(\mu, \sigma^2)$

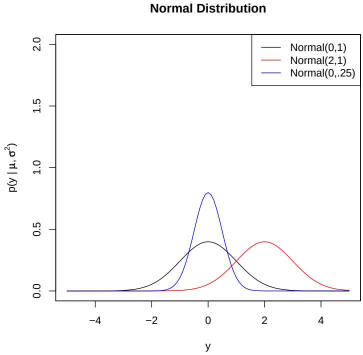
$y \in \mathbb{R}$

mean: $\mu \in \mathbb{R}$
variance: $\sigma^2 > 0$

$$p(y|\mu, \sigma^2) = \frac{\exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right)}{\sigma\sqrt{2\pi}}$$

$E(Y) = \mu$

$\text{Var}(Y) = \sigma^2$



The Multivariate Normal Distribution

$Y \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$

$\mathbf{y} \in \mathbb{R}^k$

mean vector: $\boldsymbol{\mu} \in \mathbb{R}^k$
variance-covariance matrix: $\boldsymbol{\Sigma}$ positive definite $k \times k$ matrix

$p(\mathbf{y}|\boldsymbol{\mu}, \boldsymbol{\pi}) = (2\pi)^{-k/2}|\boldsymbol{\Sigma}|^{-1/2} \exp\left(-\frac{1}{2}(\mathbf{y} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{\mathbf{-1}}(\mathbf{y} - \boldsymbol{\mu})\right)$

$E(Y) = \boldsymbol{\mu}$

$\mathrm{Var}(Y) = \boldsymbol{\Sigma}$

The Uniform Distribution

$Y \sim \text{Uniform}(\alpha, \beta)$

$y \in [\alpha, \beta]$

Interval: $[\alpha, \beta]; \;\; \beta > \alpha$

$p(y|\alpha, \beta) = \frac{1}{\beta - \alpha}$

$E(Y) = \frac{\alpha + \beta}{2}$

$\text{Var}(Y) = \frac{(\beta - \alpha)^2}{12}$

The Beta Distribution

$Y \sim \text{Beta}(\alpha, \beta)$

$y \in [0, 1]$

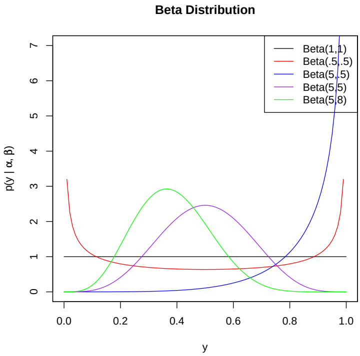
shape parameters:

$\alpha > 0; \quad \beta > 0$

$p(y|\alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} y^{(\alpha-1)}(1-y)^{(\beta-1)}$

$E(Y) = \frac{\alpha}{\alpha+\beta}$

$\text{Var}(Y) = \frac{\alpha\beta}{(\alpha+\beta)^2)\alpha+\beta+1)}$



The Gamma Distribution

$Y \sim \text{Gamma}(\alpha, \beta)$

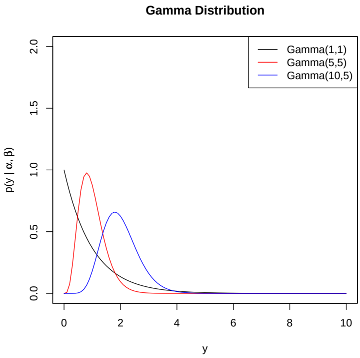
$y > 0$

shape parameter: $\alpha > 0$
inverse scale parameter: $\beta > 0$

$p(y|\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} y^{(\alpha-1)} \exp\left(-\beta y\right)$

$E(Y) = \frac{\alpha}{\beta}$

$\text{Var}(Y) = \frac{\alpha}{\beta^2}$



The Inverse Gamma Distribution

Distribution of the Inverse of a Gamma Distribution: If $X \sim \text{Gamma}(\alpha, \beta)$, then $\frac{1}{X} \sim \text{Invgamma}(\alpha, \beta)$.

$Y \sim \text{Invgamma}(\alpha, \beta)$

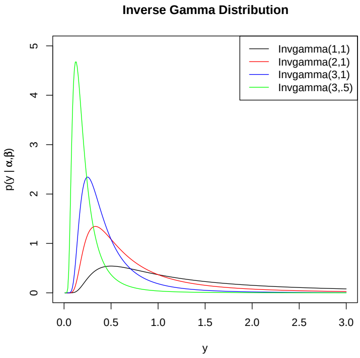
$y > 0$

shape parameter: $\alpha > 0$
scale parameter: $\beta > 0$

$p(y|\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} y^{-(\alpha+1)} e^{-\frac{\beta}{y}}$

$E(Y) = \frac{\beta}{\alpha - 1}$ for $\alpha > 1$

$\text{Var}(Y) = \frac{\beta^2}{(\alpha-1)^2(\alpha-2)}$ for $\alpha > 2$



The Dirichlet Distribution

$Y \sim \text{Dirichlet}(\alpha_1, \ldots, \alpha_k)$

$y_j \in [0, 1]; \;\; \sum_{j=1}^{k} y_j = 1$

α parameters: $\alpha_j > 0; \;\; \sum_{j=1}^{k} \alpha_j \equiv \alpha_0$

$p(\mathbf{y}|\boldsymbol{\alpha}) = \frac{\Gamma(\alpha_1 + \cdots + \alpha_k)}{\Gamma(\alpha_1) \ldots \Gamma(\alpha_k)} y_1^{\alpha_1 - 1} \ldots y_k^{\alpha_k - 1}$

$E(Y_j) = \frac{\alpha_j}{\alpha_0}$

$\text{Var}(Y_j) = \frac{\alpha_j(\alpha_0 - \alpha_j)}{\alpha_0^2(\alpha_0 + 1)}$

$\text{Cov}(Y_i, Y_j) = -\frac{\alpha_i \alpha_j}{\alpha_0^2(\alpha_0 + 1)}$