Week 1 Problems

1. An urn contains 10 red balls and 15 white balls. You pick two balls at random without replacement.

   a) What is the probability that the first ball is red?
   b) What is the probability that the second ball is red?
   c) What is the probability that both balls are white?
   d) What is the probability that the second ball is red given that the first ball is white?
   e) What is the probability that the first ball is red given that the second ball is white?

2. (From Gelman 3.7) A student sits on a street corner for an hour and records the number of bicycles b and the number of other vehicles v that go by. Two models are considered:

   - The outcomes b and v have independent Poisson distributions, with unknown means θ_b and θ_v.
   - The outcome b has a binomial distribution, with unknown probability p and sample size $b+v$.

   Show that the two models have the same likelihood if we define $p = \frac{\theta_b}{\theta_b+\theta_v}$.

   Hints:

   - Find the conditional distribution of b conditioning on information you know.
   - If $X \sim \text{Poisson}(\theta_1)$ and $Y \sim \text{Poisson}(\theta_2)$, then $X + Y \sim \text{Poisson}(\theta_1 + \theta_2)$.

3. Let $X \sim \text{Uniform}(1{,}4)$. Use calculus to find $E(X)$ and $\text{Var}(X)$.

4. a) Suppose you have n independent observations X_i from an exponential distribution where

   $$p(x_i|\lambda) = \lambda e^{-\lambda x_i}$$

   Analytically find the maximum likelihood estimate of λ.

   b) Now reparameterize the distribution for X_i in terms of τ where

   $$\tau = \frac{1}{\lambda}$$

   Find the MLE for τ.

5. Suppose that X follows a Gamma(α, β) distribution. Show that $\frac{1}{X}$ follows an Inv-Gamma(α, β) distribution.

   - Gamma PDF: $p(y) = \frac{\beta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\beta y}$
   - Inverse Gamma PDF: $p(y) = \frac{\beta^\alpha}{\Gamma(\alpha)} y^{-(\alpha+1)} e^{-\beta/y}$
   - Change of Variables formula: Let $Y = g(X)$ and $X = g^{-1}(Y)$. Then

   $$p_Y(y) = p_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$$