Week 1 Problems

1. An urn contains 10 red balls and 15 white balls. You pick two balls at random without replacement.

   Let E be the event that the first ball is red and let F be the event that the second ball is red.

   a) What is the probability that the first ball is red?

      $P(E) = \frac{10}{25} = \mathbf{\frac{2}{5}}$

   b) What is the probability that the second ball is red?

      $P(F) = P(F|E)P(E) + P(F|\bar{E})P(\bar{E}) = \left(\frac{9}{24}\right)\left(\frac{10}{25}\right) + \left(\frac{10}{24}\right)\left(\frac{15}{25}\right) = \frac{240}{600} = \mathbf{\frac{2}{5}}$

   c) What is the probability that both balls are white?

      $P(\bar{E} \cap \bar{F}) = P(\bar{F}|\bar{E})P(\bar{E}) = \left(\frac{14}{24}\right)\left(\frac{15}{25}\right) = \frac{210}{600} = \mathbf{\frac{7}{20}}$

   d) What is the probability that the second ball is red given that the first ball is white?

      $P(F|\bar{E}) = \frac{10}{24} = \mathbf{\frac{5}{12}}$

   e) What is the probability that the first ball is red given that the second ball is white?

      $$P(E|\bar{F}) = \frac{P(E \cap \bar{F})}{P(\bar{F})} = \frac{P(\bar{F}|E)P(E)}{P(\bar{F})} = \frac{\left(\frac{15}{24}\right)\left(\frac{2}{5}\right)}{1 - \frac{2}{5}} = \frac{\left(\frac{15}{24}\right)\left(\frac{2}{5}\right)}{\frac{3}{5}} = \frac{\frac{30}{120}}{\frac{3}{5}} = \mathbf{\frac{5}{12}}$$

2. (From Gelman 3.7) A student sits on a street corner for an hour and records the number of bicycles b and the number of other vehicles v that go by. Two models are considered:

   - The outcomes b and v have independent Poisson distributions, with unknown means θ_b and θ_v.
   - The outcome b has a binomial distribution, with unknown probability p and sample size $b+v$.

   Show that the two models have the same likelihood if we define $p = \frac{\theta_b}{\theta_b+\theta_v}$.

   Hints:

   - Find the conditional distribution of b conditioning on information you know.
   - If $X \sim \text{Poisson}(\theta_1)$ and $Y \sim \text{Poisson}(\theta_2)$, then $X + Y \sim \text{Poisson}(\theta_1 + \theta_2)$.

   We are given that the total number of bicycles and vehicles is $b + v$. We are also told that the likelihood of b is a binomial distribution. So we can show that the likelihood of b is a binomial by starting from the two Poisson distributions and conditioning on the total number of vehicles $b+v$. Thus, the likelihood we are trying to find is $p(b|b+v)$. We need to use Bayes' Rule for conditional probability distributions.

   $$\begin{aligned} p(b|b+v) &= \frac{p(b+v|b)p(b)}{p(b+v)} \\ &= \frac{p(v)p(b)}{p(b+v)} \\ &= \frac{\text{Poisson}(\theta_v)\ \text{Poisson}(\theta_b)}{\text{Poisson}(\theta_b+\theta_v)} \end{aligned}$$

The denominator is a Poisson($\theta_b + \theta_v$) distribution because the sum of Poisson distributions is also a Poisson distribution.

We can then do some math to simplify.

$$\begin{aligned}
p(b|b+v) &= \frac{\frac{e^{-\theta_v}\theta_v^v}{v!}\frac{e^{-\theta_b}\theta_b^b}{b!}}{\frac{e^{-(\theta_b+\theta_v)}(\theta_b+\theta_v)^{b+v}}{(b+v)!}} \\
&= \frac{(b+v)!}{b!\ v!}\frac{\theta_b^b\theta_v^v}{(\theta_b+\theta_b)^{b+v}} \\
&= \frac{(b+v)!}{b!\ v!}\frac{\theta_b^b}{(\theta_b+\theta_v)^b}\frac{\theta_v^v}{(\theta_b+\theta_v)^v} \\
&= \frac{(b+v)!}{b!\ v!}\left(\frac{\theta_b}{\theta_b+\theta_v}\right)^b\left(\frac{\theta_v}{\theta_b+\theta_v}\right)^v \\
&= \frac{(b+v)!}{b!\ v!}\left(\frac{\theta_b}{\theta_b+\theta_v}\right)^b\left(\frac{\theta_v+\theta_b-\theta_b}{\theta_b+\theta_v}\right)^v \\
&= \frac{(b+v)!}{b!\ v!}\left(\frac{\theta_b}{\theta_b+\theta_v}\right)^b\left(1-\frac{\theta_b}{\theta_b+\theta_v}\right)^v
\end{aligned}$$

This is a binomial distribution with $b+v$ trials and probability $\frac{\theta_b}{\theta_b+\theta_v}$.

3. Let $X \sim$ Uniform(1,4). Use calculus to find $E(X)$ and Var(X).

   Let X be distributed uniform over the interval (1,4). We know the formula for the variance of a random variable:

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$

   To find $E(X)$, we simply take the integral of the form

$$\int_1^4 xp(x)\ dx$$

   where $f(x)$ is the PDF of the uniform density. We can think of this as analogous to the discrete case, where we take the sum of each x value weighted by its probability. So we end up with

$$\begin{aligned}
E(X) &= \int_1^4 xp(x)\ dx \\
&= \int_1^4 x\frac{1}{4-1}\ dx \\
&= \frac{1}{3}\int_1^4 x\ dx \\
&= \frac{1}{3}\frac{1}{2}x^2\Big|_1^4 \\
&= \frac{16}{6} - \frac{1}{6} \\
&= \mathbf{\frac{15}{6}}
\end{aligned}$$

   To find $E(X^2)$, we take the following integral

$$\int_1^4 x^2 p(x)\, dx$$

using what is sometimes known as the *Law of the Unconscious Statistician*. Basically, we can find the expected value of a function of X by using the PDF of X.

$$\begin{aligned}
E(X^2) &= \int_1^4 x^2 p(x)\, dx \\
&= \frac{1}{3}\int_1^4 x^2\, dx \\
&= \frac{1}{3}\frac{1}{3}x^3\Big|_1^4 \\
&= \frac{64}{9} - \frac{1}{9} \\
&= \frac{63}{9} \\
&= 7
\end{aligned}$$

The variance is then simply

$$\begin{aligned}
\mathrm{Var}(X) &= E(X^2) - [E(X)]^2 \\
&= 7 - \left(\frac{15}{6}\right)^2 \\
&= 7 - \frac{225}{36} \\
&= 7 - \frac{25}{4} \\
&= \mathbf{\frac{3}{4}}
\end{aligned}$$

4. a) Suppose you have n independent observations X_i from an exponential distribution where

$$p(x_i|\lambda) = \lambda e^{-\lambda x_i}$$

Analytically find the maximum likelihood estimate of λ.

$$\begin{aligned}
L(\lambda|X) &= \prod_{i=1}^n \lambda e^{-\lambda x_i} \\
l(\lambda|X) &= \sum_{i=1}^n \log\lambda - \lambda x_i \\
&= n\log\lambda - \lambda\sum_{i=1}^n x_i \\
l'(\lambda|X) &= \frac{n}{\lambda} - \sum_{i=1}^n x_i
\end{aligned}$$

Setting the derivative to 0 and solving for $\hat{\lambda}$:

$$\begin{aligned} 0 &= \frac{n}{\lambda} - \sum_{i=1}^{n} x_i \\ \hat{\lambda} &= \frac{n}{\sum_{i=1}^{n} x_i} \end{aligned}$$

b) Now reparameterize the distribution for X_i in terms of τ where

$$\tau = \frac{1}{\lambda}$$

Find the MLE for τ.

$$\begin{aligned} L(\tau|X) &= \prod_{i=1}^{n} \frac{1}{\tau} e^{-\frac{x_i}{\tau}} \\ l(\tau|X) &= \sum_{i=1}^{n} -\log \tau - \frac{x_i}{\tau} \\ &= -n \log \tau - \frac{\sum_{i=1}^{n} x_i}{\tau} \\ l'(\tau|X) &= -\frac{n}{\tau} + \frac{\sum_{i=1}^{n} x_i}{\tau^2} \end{aligned}$$

Setting the derivative to 0 and solving for $\hat{\tau}$:

$$\begin{aligned} 0 &= -\frac{n}{\tau} + \frac{\sum_{i=1}^{n} x_i}{\tau^2} \\ \frac{n}{\tau} &= \frac{\sum_{i=1}^{n} x_i}{\tau^2} \\ n &= \frac{\sum_{i=1}^{n} x_i}{\tau} \\ \hat{\tau} &= \frac{\sum_{i=1}^{n} x_i}{n} \end{aligned}$$

5. Suppose that X follows a Gamma(α, β) distribution. Show that $\frac{1}{X}$ follows an Inv-Gamma(α, β) distribution.

- Gamma PDF: $p(y) = \frac{\beta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\beta y}$
- Inverse Gamma PDF: $p(y) = \frac{\beta^\alpha}{\Gamma(\alpha)} y^{-(\alpha+1)} e^{-\beta/y}$
- Change of Variables formula: Let $Y = g(X)$ and $X = g^{-1}(Y)$. Then

$$p_Y(y) = p_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$$

So let $Y = \frac{1}{X}$ and $X = \frac{1}{Y}$. If X follows a Gamma distribution, then we need to show that Y follows an Inverse Gamma distribution.

$$
\begin{aligned}
p_Y(y) &= p_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)} g^{-1}(y)^{\alpha-1} e^{-\beta g^{-1}(y)} \left| -\frac{1}{y^2} \right| \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)} \left(\frac{1}{y} \right)^{\alpha-1} e^{\frac{-\beta}{y}} \left(\frac{1}{y^2} \right) \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)} y^{-(\alpha-1)} e^{\frac{-\beta}{y}} y^{-2} \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)} y^{-(\alpha+1)} e^{\frac{-\beta}{y}}
\end{aligned}
$$