Week 2 Problems

1. Suppose you have n independent observations that follow a Poisson(λ), so

$$Y_i \sim \text{Poisson}(\lambda) \text{ for } i = 1, 2, \ldots, n$$

Using a Gamma(α, β) prior on λ, find the posterior distribution for λ. Does this distribution have a name?

- Poisson PMF: $p(y) = \frac{e^{-\lambda}\lambda^y}{y!}$
- Gamma PDF: $p(y) = \frac{\beta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\beta y}$

$$\begin{aligned}
p(\lambda|\mathbf{y}) &\propto p(\mathbf{y}|\lambda)p(\lambda) \\
&= \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{y_i}}{y_i!} \times \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda} \\
&\propto \prod_{i=1}^{n} e^{-\lambda}\lambda^{y_i} \times \lambda^{\alpha-1} e^{-\beta\lambda} \\
&= e^{-n\lambda-\beta\lambda} \lambda^{\sum_{i=1}^{n} y_i+\alpha-1} \\
&= e^{-\lambda(n+\beta)} \lambda^{\sum_{i=1}^{n} y_i+\alpha-1}
\end{aligned}$$

$$p(\lambda|\mathbf{y}) \propto e^{-\lambda(n+\beta)} \lambda^{\sum_{i=1}^{n} y_i+\alpha-1}$$

This looks like the kernel for a Gamma distribution. Turns out the posterior is a Gamma($\sum_{i=1}^{n} y_i + \alpha, n + \beta$) distribution.